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Probabilistic Unawareness

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Probabilistic Unawareness[1]

Mikaël Cozic[2]

Abstract

Awareness and unawareness are a major theme in the epistemic logic literature, but it is handled there only in terms in terms of full belief operators. The present paper aims at a treatment in terms of partial belief operators. It draws upon the modal probabilistic logic that was introduced by Aumann (1999) at the semantic level and then axiomatized by Heifetz and Mongin (2001). The paper embodies in this framework those properties of unawareness which have been highlighted in the seminal paper by Modica and Rustichini (1999). This last paper is concerned with full belief, but we argue that the properties in question also apply to partial belief in the chosen probabilistic sense. The main result is a (soundness and) completeness theorem that reunites the two strands, modal and probabilistic, of epistemic logic.

Keywords

Unawareness, epistemic logic, probabilistic logic.

Résumé

La conscience et l'absence de conscience sont devenus des thèmes majeurs en logique épistémique, mais elles n'y sont traitées qu'avec des opérateurs de croyance catégorique. L'article entend considérer des opérateurs de croyance partielle. Il s'appuie sur une logique modale probabiliste introduite par Aumann (1999) au niveau sémantique et axiomatisée ensuite par Heifetz et Mongin (2001). L'article introduit dans ce cadre les propriétés de l'absence de conscience mises en avant par Modica et Rustichini (1999) dans leur article fondateur. Celui-ci n'aborde que la croyance catégorique, mais nous faisons valoir que les propriétés en question s'appliquent également aux croyances partielles au sens probabiliste que l'on a choisi. Le résultat principal est un théorème de complétude qui réunit les deux domaines, modal et probabiliste, de la logique épistémique.

Mots-clés

Unawareness, epistemic logic, probabilistic logic.

Classification JEL

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1 Introduction

Full (or, categorical) beliefs are epistemic attitudes like those ascribed when one says

Pierre believes that ϕ

The main formalism of full beliefs is currently provided by epistemic logic. It is well-known that it suffers from two main cognitive idealizations. The first one is *logical omniscience*: a family of properties such as closure of beliefs under logical consequence (from the premiss that ϕ implies ψ, infer that $B\phi$ implies $B\psi$, Rule of Monotonicity) or substituability of logically equivalent formulas (from the premiss that ϕ is equivalent to ψ, infer that $B\phi$ is equivalent to $B\psi$, Rule of Equivalence). The second cognitive idealization is *full awareness*, which is more difficult to characterize precisely. As a first approximation, the agent is supposed to have a full understanding of the underlying state space.

Much attention has been devoted to the weakening of logical omniscience. Fagin, Halpern, Moses & Vardi (1995) survey this literature and in particular its two main current solutions: structures with subjective, logically impossible states[1] and awareness© structures (Fagin & Halpern 1988)[2]. The very same awareness© structures are used to weaken the full awareness assumption. More recently, game theorists became interested in weakening awareness (Dekel, Lipman & Rustichini (1998), Modica & Rustichini (1999), Heifetz, Meier & Schipper (2006)).

Modal epistemic logic is a rather coarse-grained model of doxastic attitudes, because it excludes *partial beliefs*, i.e. the fact that an agent believes that it is unlikely that ϕ or very likely that ψ. The main formalism for partial beliefs makes use of probabilities in their subjective or epistemic interpretation, where probability values stand for degrees of belief. There is a noteworthy contrast between modal epistemic logic and those probabilistic models: whereas the former make beliefs explicit (part of the formal language), they are left implicit in the latter. However, Aumann (1999) has introduced in the syntax partial belief operators interpretable as "the agent believes at least to degree α that...", and he has proposed a possible-world semantics for them (which is inspired by Harsanyi (1967)). This semantics has been axiomatized by Heifetz & Mongin (2001) under the form of a weak (soundness and) completeness theorem. This *probabilistic logic* is the true counterpart of Kripkean epistemic logic for degrees of beliefs, and it is the framework of this paper.

This logic suffers from the same cognitive idealizations: logical omniscience and full awareness. A preceding paper (Cozic 2007) dealt with the problem of logical omniscience in probabilistic logic. Our proposal was mainly based on the use of so-called *impossible states* i.e. subjective states where the logical connectives can have a non-classical behavior. The aim of the present paper is to enrich probabilistic logic with a modal logic of unawareness. Our main proposal is a generalization of Aumman's semantics that uses impossible states like those of Modica & Rustichini (1999), and provably satisfies a list of intuitive

[1] Hintikka (1975) and Wansing (1990)

[2] In the whole paper we use "awareness©" to denote the *model* of (Fagin & Halpern 1988), to be distinguished from the *attitude* of awareness.

requirements. Our main result is a weak completeness theorem like the one demonstrated by Heifetz & Mongin (2001), but adapted to the richer framework that includes awareness.

The remainder of the paper proceeds as follows. In Section 2, we try to provide some intuitions about the target attitudes, awareness and unawareness. Section 3 presents briefly probabilistic logic, and notably Heifetz & Mongin (2001)'s axiom system (named 'System HM'). In Section 4, we vindicate a slightly modified version of Modica & Rustichini (1999)'s Generalized Standard Structures. Section 5 contains the main contribution of the paper: a logic for dealing with unawareness in probabilistic logic. Our axiom system (named 'system HM_U') enriches the probabilistic logic with an awareness operator and accompanying axioms. Section 6 concludes.

2 Awareness and unawareness

2.1 Some intuitions

Since unawareness is a more elusive concept than logical omniscience. This section gives insights on the target phenomena and puts forward properties that a satisfactory logic of (un)awareness should embody. Following the lead of Modica & Rustichini (1999), we may say that there is unawareness when

- there is "ignorance about the state space"
- "some of the facts that determine which state of nature occurs are not present in the subject's mind"
- "the agent does not know, does not know that she does not know, does not know that she does not know that she does not know, and so on..."

Here is an illustrative example. Pierre plans to rent a house for the next holiday, and, from the observer's point of view, there are three main factors relevant to his choice:

- p: the house is no more than 1 km far from the sea
- q: the house is no more than 1 km far from a bar
- r: the house is no more than 1 km far from an airport

There is an intuitive distinction between the two following doxastic states:

state (i): Pierre simply ignores whether it is true that r : he does not know[3] whether there is an airport no more than 1 km away from the house - there are both r-states and $\neg r$-states which are epistemically accessible to him.

[3]We use sometimes "know" rather than "believe", but here and in the sequel, we deal with belief and not knowledge.

state (ii): the possibility that r does not come up to Pierre's mind, so Pierre cannot ask himself : 'is there an airport no more than 1 km far from the house?".

The contrast between the two epistemic states can be rendered in terms of a state space with either fine or coarse grain[4]. The observer's set of possible states is:

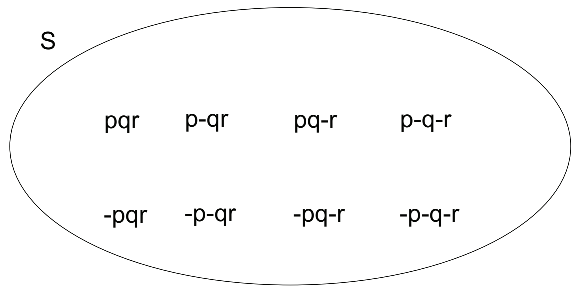


Figure 1: An objective state space

<INSERT HERE FIGURE 1>
where each state is labeled by the sequence of literals that are true in it. This state is also Pierre's in doxastic state (i). The doxastic state (ii), on the other hand, is:

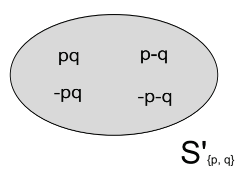


Figure 2: A subjective state space

<INSERT HERE FIGURE 2>
Some states in the initial state space have been fused with each other - those which differ only in the truth value they assign to the formula the agent is unaware of, namely r.[5]

[4] This close to the "small world" concept of Savage (1954). In Savage's language, "world" means the state space or set of possible worlds, itself.

[5] Once again, the idea is already present in Savage (1954): "...a smaller world is derived from a larger by neglecting some distinctions between states".

2.2 Some principles in epistemic logic

More theoretically, what properties should one expect awareness to satisfy ? In what follows

- $B\phi$ means "the agent believes that ϕ,
- $A\phi$ means "the agent is aware that ϕ", and

$A\phi \leftrightarrow A\neg\phi$	(Symmetry)
$A(\phi \wedge \psi) \leftrightarrow A\phi \wedge A\psi$	(Distributivity over $\wedge$)
$A\phi \leftrightarrow AA\phi$	(Self-Reflection)
$\neg A\phi \rightarrow \neg A\neg A\phi$	(U-introspection)
$\neg A\phi \rightarrow \neg B\phi \wedge \neg B\neg B\phi$	(Plausibility)
$\neg A\phi \rightarrow (\neg B)^n\phi \ \forall n \in \mathbb{N}$	(Strong Plausibility)
$\neg B\neg A\phi$	(BU-introspection)

Natural as they are, these properties cannot be jointly satisfied in Kripkean epistemic logic. This has been recognized by Dekel et al. (1998), who show that it is impossible to have both

(i) a non-trivial awareness operator which satisfies Plausibility, U-introspection and BU-introspection, and

(ii) a belief operator which satisfies either Necessitation or the Rule of Monotonicity.

Of course, the standard belief operator of epistemic logic *does* satisfy both Necessitation and the Rule of Monotonicity. The main challenge is therefore to build a logic of belief *and* awareness that supports the above intuitive principles. Since Necessitation and the Rule of Monotonicity are nothing but forms of logical omniscience, it becomes a major prerequisite to weaken latter. Indeed, both the Generalized Standard Structures of Modica & Rustichini (1999) and the awareness© structures of Fagin & Halpern (1988) do weaken logical omniscience.

2.3 Some principles in probabilistic logic

Probabilistic logics are both lesser known than, and not so well unified as modal epistemic logics. The syntactic framework, in particular, varies from one to another. The logic on which this paper is based relies on a language which is quite similar to that of epistemic logic, and it therefore can be seen as constituting a probabilistic *modal* logic: Its primary doxastic operators are L_a, where a is a rational number between 0 and 1 ("the agent believes at least to degree a that...")[6] We can express the relevant intuitive principles for L_a as:

$A\phi \leftrightarrow A\neg\phi$	(Symmetry)
$A(\phi \wedge \psi) \leftrightarrow A\phi \wedge A\psi$	(Distributivity over $\wedge$)
$A\phi \leftrightarrow AA\phi$	(Self-Reflection)
$\neg A\phi \rightarrow \neg A\neg A\phi$	(U-introspection)
$\neg A\phi \rightarrow \neg L_a\phi \wedge \neg L_a\neg L_a\phi$	(Plausibility)
$\neg A\phi \rightarrow (\neg L_a)^n\phi \ \forall n \in \mathbb{N}$	(Strong Plausibility)
$\neg L_a\neg A\phi$	(L_aU-introspection)
$L_0\phi \leftrightarrow A\phi$	(Minimality)

Seven of these eight principles are direct counterparts of those put forward for modal epistemic logic, Minimality being the exception. On the one hand, if an agent believes to some degree (however small) that ϕ, then he or she is aware of ϕ. This conditional is intuitive for a judgmental rather than for a purely behavioral conception of partial beliefs. Vickers (1976) philosophically elaborates on this distinction, in relationship with probabilities. The reverse conditional roughly means that an agent aware of ϕ has some degree of belief toward ϕ. This directly echoes Bayesian epistemology. These eight principles may be seen as a set of requirements for a satisfactory probabilistic logic.

3 Probabilistic (modal) logic

This section briefly reviews the main concepts of probabilistic logic following Aumann (1999) and Heifetz & Mongin (2001)[7]. Probabilistic logic is of course related to the familiar models of beliefs where the doxastic states are represented by a probability distribution on a state space (or on the formulas of a propositional language), but the doxastic operator is made explicit here. Syntactically, as we have already said, this means that the language is endowed with a family of operators L_a. Semantically, there are sets of possible states (or "events") corresponding to the fact that an agent believes (or does not believe) such and such formula at least to such and such degree.

3.1 Language

Definition 1 (probabilistic language)
The set of formulas of a probabilistic language $\mathcal{L}^L(At)$ based on a set At of propositional variables is defined by :

[6] By contrast, Fagin & Halpern (1991) and Halpern (2003) allow formulas like $a_1w(\phi_1)+...+a_nw(\phi_n) \geq b$ where $a_1, ..., a_n, b$ are integers, $\phi_1, ..., \phi_n$ are propositional formulas and $w(\phi)$ is to be interpreted as the probability of ϕ.

[7] Economists are leading contributors to the study of explicit probabilistic structures because they correspond to the so-called *type spaces*, which are basic to games of incomplete information since Harsanyi (1967). See Aumann & Heifetz (2002).

$$\phi ::= p|\bot|\top|\neg\phi|\phi \wedge \phi|L_a\phi$$

where $p \in At$ and $a \in [0,1] \cap \mathbb{Q}$.

From this, one may define two derived belief operators:

- $M_a\phi = L_{1-a}\neg\phi$ (the agent believes *at most* to degree a that ϕ)
- $E_a\phi = M_a\phi \wedge L_a\phi$ (the agent believes *exactly* to degree a that ϕ)[8]

3.2 Semantics

Probabilistic structures (PS), as introduced by Aumann (1999) to interpret the above formal language. They are the true probabilistic counterpart of Kripke structures for epistemic logic. In particular, iterated beliefs are allowed because a probability distribution is attributed to *each* possible state, very much like a Kripke relation. We follow Heifetz & Mongin (2001)'s definition:

Definition 2 (probabilistic structures)
A ***probabilistic structure*** *for $\mathcal{L}^L(At)$ is a 4-uple $\mathcal{M} = (S, \Sigma, \pi, P)$ where*

(i) S is a state space

(ii) Σ is a σ-field of subsets of S

(iii) $\pi : S \times At \to \{0,1\}$ is a valuation for S s.t. $\pi(., p)$ is measurable for every $p \in At$

(iv) $P : S \to \Delta(S, \Sigma)$ is a measurable mapping from S to the set of probability measures on Σ endowed with the σ-field generated by the sets
$\{\mu \in \Delta(S,\Sigma) : \mu(E) \geq a\}$ $\forall E \in \Sigma$, $a \in [0,1]$.

Definition 3
The ***satisfaction relation****, labelled $\vDash$, extends π to every formula of the language according to the following conditions :*

(i) $\mathcal{M}, s \vDash p$ iff $\pi(p, s) = 1$

(ii) $\mathcal{M}, s \vDash \phi \wedge \psi$ iff $\mathcal{M}, s \vDash \phi$ and $\mathcal{M}, s \vDash \psi$

(iii) $\mathcal{M}, s \vDash \neg\phi$ iff $\mathcal{M}, s \nvDash \phi$

[8] Since a is a rational number and the structures will typically include real-valued probability distributions, it may happen in some state that for no a it is true that $E_a\phi$. It happens when the probability assigned to ϕ is a real but non-rational number.

(iv) $\mathcal{M}, s \models L_a\phi$ *iff* $P(s)([[\phi]]) \geq a$

As usual, $[[\phi]]$ denotes the set of states where ϕ is true - or the proposition expressed by ϕ. From a logical point of view, one of the most striking features of probabilistic structures is that *compactness does not hold*. Let $\Gamma = \{L_{1/2-1/n}\phi : n \geq 2, n \in \mathbb{N}\}$ and $\psi = \neg L_{1/2}\phi$. One should expect semantically that, for each finite $\Gamma' \subset \Gamma$, $\Gamma' \cup \{\psi\}$ is satisfiable, but $\Gamma \cup \{\psi\}$ is not. As a consequence, an axiomatization of probabilistic structures will provide at best a *weak* completeness theorem.

3.3 Axiomatization

Aumann (1999) having failed to axiomatize explicit probabilistic structures, Heifetz & Mongin (2001) devised an axiom system that is (weakly) complete for his structures. We coin it the *System HM*[9].

System HM

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(L1) $L_0\phi$
(L2) $L_a\top$
(L3) $L_a\phi \rightarrow \neg L_b\neg\phi$ $(a + b > 1)$
(L4) $\neg L_a\phi \rightarrow M_a\phi$
(Def M) $M_a\phi \leftrightarrow L_{1-a}\neg\phi$
(RE) From $\phi \leftrightarrow \psi$ infer $L_a\phi \leftrightarrow L_a\psi$

(B) From $((\phi_1, ..., \phi_m) \leftrightarrow (\psi_1, ..., \psi_n))$ infer
$((\bigwedge_{i=1}^{m} L_{ai}\phi_i) \wedge (\bigwedge_{j=2}^{n} M_{bj}\psi_j) \rightarrow L_{(a1+...+am)-(b1+...+bn)}\psi_1)$

The inference rule (B) deserves attention. Heifetz & Mongin (2001) explains the content and the origin of (B), so we can be brief. The pseudo-formula $((\phi_1, ..., \phi_m) \leftrightarrow (\psi_1, ..., \psi_n))$ is an abbreviation for

$$\bigwedge_{k=1}^{max(m,n)} \phi^{(k)} \leftrightarrow \psi^{(k)}$$

where

$$\phi^{(k)} = \bigvee_{1 \leq l_1 \leq ... \leq l_k < m} (\phi_{l_1} \wedge ... \wedge \phi_{l_k})$$

[9]Heifetz & Mongin (2001) call this system $\Sigma+$.

(if $k > m$, by convention $\phi^{(k)} = \bot$). Intuitively, $\phi^{(k)}$ "says" that at least k of the formulas ϕ_i are true. The meaning of (B) is simpler to grasp when it is interpreted set-theoretically. Associate $(E_1, ..., E_m)$ and $(F_1, ..., F_n)$, two sequences of events, to the sequences of formulas $(\phi_1, ..., \phi_m)$ and $(\psi_1, ..., \psi_n)$. Then, the premiss of (B) is a syntactical rendering of the idea that the sum of the characteristic functions is equal, i.e. $\sum_{i=1}^{m} \mathbf{I}_{E_i} = \sum_{j=1}^{n} \mathbf{I}_{F_j}$. If $P(E_i) \geq \alpha_i$ for $i = 1, ..., n$ and $P(F_j) \leq \beta_j$ for $j = 2, ..., m$, then $P(F_1)$ has to "compensate", i.e.

$$P(F_1) \geq (\alpha_1 + ... + \alpha_n) - (\beta_2 + ... + \beta_m)$$

The conclusion of (B) is a translation of this "compensation". It is very powerful from the probabilistic point of view, and plays a crucial role in the (sophisticated) completeness proof. In comparison with modal epistemic logic, one of the issues is that it is not easy to adapt the usual proof method, *i.e.* that of canonical models. More precisely, with Kripke logics, there is a natural accessibility relation on the canonical state space. Here, we need to prove the existence of a canonical probability distribution from relevant mathematical principles. This step is linked to a difficulty that is well-known in the axiomatic study of quantitative and qualitative probability: how to ensure a (numerical) probabilistic representation for a finite structure of qualitative probability ? A principle similar to (B) has been introduced in an set-theoretical axiomatic framework by Kraft, Pratt & Seidenberg (1959) and imported into a probabilistic modal logic (with a qualitative binary operator) by Gärdenfors (1975).

4 A detour by epistemic logic without full awareness

Our probabilistic logic without full awareness is largely an adaptation of the Generalized Standard Structures (GSS) introduced by Modica & Rustichini (1999) to deal with unawareness in epistemic logic. Actually, we will slightly modify Modica & Rustichini (1999)'s semantics and obtain a partial semantics for unawareness. Before giving our own logical system, we remind the reader of GSSs of the case of epistemic logic.

4.1 Basic heuristics

Going back to the motivating example, suppose that the "objective" state space is based on the set of atomic formulas $At = \{p, q, r\}$ as in Figure 1. Suppose furthermore that the actual state is $s = pqr$ and that in this state, Pierre believes that p, does not know whether q and is unaware of r. Modica & Rustichini (1999) would say that the actual state is "projected" to a subjective state $\rho(s) = pq$ of a "subjective" state space based on the set of atomic formulas that the agent is aware of i.e. p and q. In Kripkean epistemic logic, the agent's *accessibility relation* selects, for each possible state s, the set of states $R(s)$ which are epistemically possible for him or her. GSSs define an accessibility relation on this subjective state space. In Figure 3, projection is represented by a dotted arrow and accessibility by solid arrows.

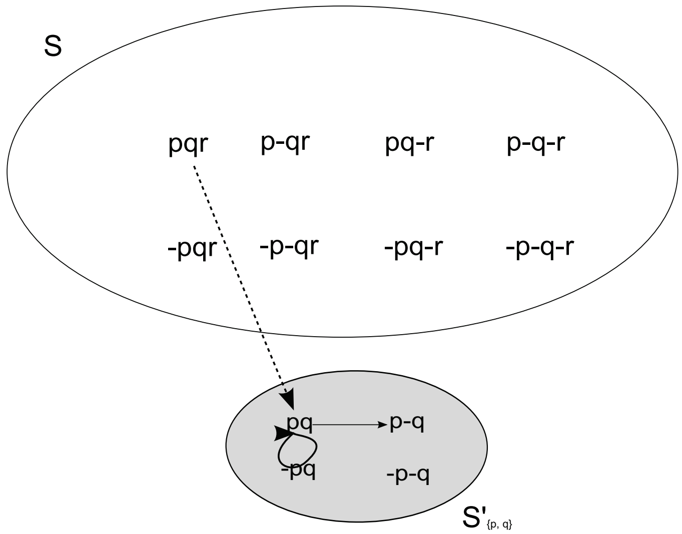


Figure 3: Projection of an objective state in a subjective state space

<INSERT HERE FIGURE 3>

This picture does not represent all projections between objective states and subjective states, and it corresponds to only one subjective state space. Generally, there are as many subjective state spaces S'_X as there are subsets X of the set At on which the objective state space is based[10]. It is crucial to specify in the right way the conditions on the projection $\rho(.)$ from objective to subjective states. Suppose that another objective state s' is projected to pq as well ; then, two conditions should obtain. First, $s = pqr$ and s' should agree on the atomic formulas the agent is aware of ; so, for instance, s' could not be $\neg pqr$, since the agent is aware of p. Second, the states accessible from s and s' them should be the same. Another natural assumption is that all the states accessible from a given state are located in the same "subjective" state space (see (v)(2) in Definition 4 below). Figure 4 pictures a GSS more faithfully than the preceding one.

<INSERT HERE FIGURE 4>

4.2 Generalized Standard Structures

The next definition is a precise rendering of the intuitive ideas. We have followed Halpern (2001*b*) rather than Modica & Rustichini (1999), notably because he makes clear that GSSs can be seen as structures with impossible states.

[10] In an objective state space, the set of states need not reflect the set of possible truth-value assignments to propositional variables (as it is the case in our example).

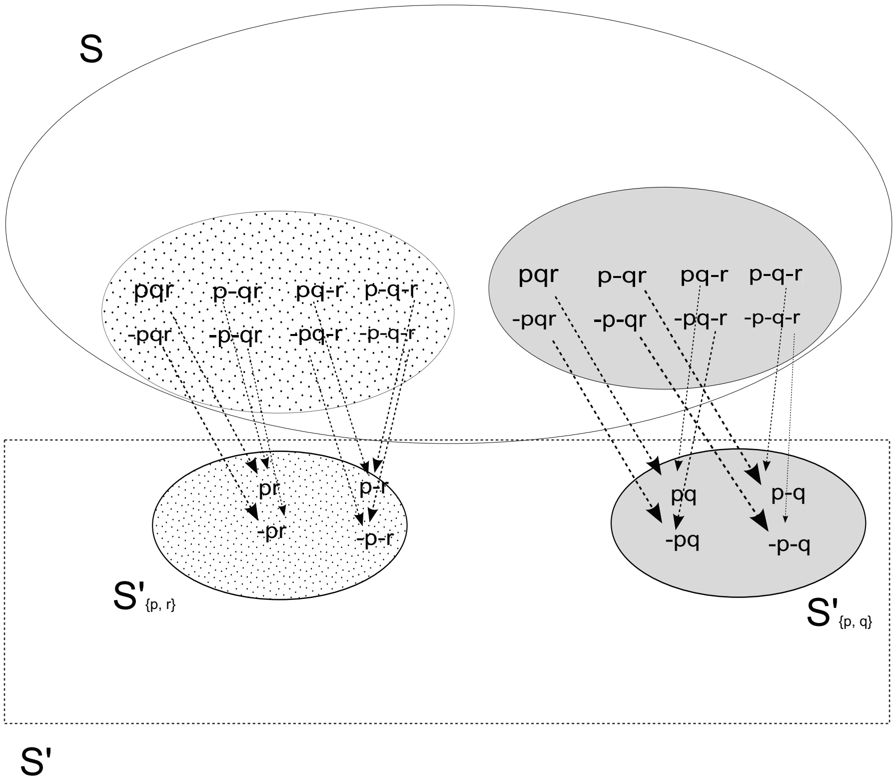


Figure 4: Partial picture of a GSS

Definition 4
A GSS is a t-uple $\mathcal{M} = (S, S', \pi, R, \rho)$

(i) S *is a state space*

(ii) $S' = \bigcup_{X \subseteq At} S'_X$ *(where* S'_X *are disjoint) is a (non-standard) state space*

(iii) $\pi : S \times At \to \{0, 1\}$ *is a valuation for* S

(iv) $R : S \to \wp(S')$ *is an accessibility relation for* S

(v) $\rho : S \to S'$ *is an onto map s.t.*

(1) if $\rho(s) = \rho(t) \in S'_X$*, then (a) for each atomic formula* $p \in X$*,* $\pi(s, p) = \pi(t, p)$ *and (b)* $R(s) = R(t)$*, and*

(2) if $\rho(s) \in S'_X$*, then* $R(s) \subseteq S'_X$

One can extend R *and* π *to the whole state space:*

(vi) $\pi' : S' \times At \to \{0, 1\}$ *is a valuation for* S' *s.t. for all* $s' \in S'_X$*,* $\pi'(s', p) = 1$ *iff (a)* $p \in X$ *and (b) for all* $s \in \rho^{-1}(s'), \pi(s, p) = 1$*. We note* $\pi^* = \pi \cup \pi'$*.*

(vii) $R' : S' \to \wp(S')$ *is an accessibility relation for* S *s.t. for all* $s' \in S'_X$, $R'(s') = R(s)$ *for some* $s \in \rho^{-1}(s')$. *We note* $R^* = R \cup R'$.

Compared with Kripke structures, a modification is required as regards negation. In a subjective state $s' \in S'_X$, for a negated formula $\neg\phi$ to be true, it has to be not only that ϕ is not true, but also that ϕ belongs to the sub-language induced by X. As a consequence, it can be the case that in some s', neither ϕ nor $\neg\phi$ is true. This is why subjective states are *impossible* states. In the sequel, $\mathcal{L}^{BA}(X)$ denotes the language containing the operators B (full beliefs) and A (awareness) and based on the set X of propositional variables.

Definition 5
The ***satisfaction relation*** *for GSS is defined for each* $s^* \in S^* = S \cup S'$:

(i) $\mathcal{M}, s^* \vDash p$ *iff* $\pi^*(s^*, p) = 1$

(ii) $\mathcal{M}, s^* \vDash \phi \wedge \psi$ *iff* $\mathcal{M}, s^* \vDash \phi$ *and* $\mathcal{M}, s^* \vDash \psi$

(iii) $\mathcal{M}, s^* \vDash \neg\phi$ *iff* $\mathcal{M}, s^* \nvDash \phi$ *and either* $s^* \in S$, *or* $s^* \in S'_X$ *and* $\phi \in \mathcal{L}^{BA}(X)$

(iv) $\mathcal{M}, s^* \vDash B\phi$ *iff for each* $t^* \in R^*(s^*)$, $\mathcal{M}, t^* \vDash \phi$

(v) $\mathcal{M}, s^* \vDash A\phi \Leftrightarrow \mathcal{M}, s^* \vDash B\phi \vee B\neg B\phi$

Example 1
In Figure 3, $\mathcal{M}, \rho(s) \nvDash r$ *given clause (vi) of the Definition of GSS. However,* $\mathcal{M}, \rho(s) \nvDash \neg r$ *given clause (iii) of the Definition of the satisfaction relation. Clause (iv) implies that* $\mathcal{M}, \rho(s) \nvDash Br$. *But clause (iii) again implies that* $\mathcal{M}, \rho(s) \nvDash \neg Br$. *The same holds in the state accessible from* $\rho(s)$. *Therefore* $\mathcal{M}, \rho(s) \nvDash B\neg Br$. *By clause (iv), this implies that* $\mathcal{M}, \rho(s) \nvDash Ar$ *and consequently that* $\mathcal{M}, s \vDash \neg Ar$.

4.3 Partial Generalized Standard Structures

The preceding definition characterizes awareness in terms of beliefs: Pierre is unaware of ϕ if, and only if, he does not believe that ϕ and does not believe that he does not believe that ϕ. This is unproblematic when one studies, as Modica & Rustichini (1999) do, *partitional* structures, i.e. structures where accessibility relation is an equivalence relation and thus a partitions of the state space. Game theorists rely extensively on this special case, which has convenient properties. In this particular case, the fact that an agent does not believe that ϕ and does not believe that she does not believe that ϕ implies that she does not believe that she does not believe...that she does not believe that ϕ - at all level of iteration[11]. But

[11] Intuitively: if the agent does not believe that ϕ, this means that not every state of the relevant partition's cell makes ϕ true. If ϕ were belonging to the sublanguage associated to the relevant subjective state space, since the accessibility relation is partitional, this would imply that $\neg B\phi$ would be true in every state of the cell. But by hypothesis, the agent does not believe that she does not believe that ϕ. We have therefore to conclude that ϕ does not belong to the sublanguage of the subjective state space (see Fact 1 below). Hence, no formula $B\neg B\neg B...\neg B\phi$ can be true.

without such an implication, the equivalence between the fact that an agent is unaware of ϕ and the fact that she does not believe that ϕ and does not believe that she does not believe that ϕ is dubious, at least in one of the two directions.

We therefore need a more general characterization of awareness and unawareness. Our proposal proceeds from the following observation: in the definition of satisfaction for GSS, the truth-conditions for negated formulas introduce (semantic) *partiality*. If $p \notin X$ and $s^* \in S'_X$ then neither $\mathcal{M}, s^* \vDash p$ nor $\mathcal{M}, s^* \vDash \neg p$ obtain. Let us indicate by $\mathcal{M}, s^* \Uparrow \phi$ that the formula ϕ is undefined at s^*, and by $\mathcal{M}, s^* \Downarrow \phi$ that it is defined. The following is true:

Fact 1
Let $\mathcal{M}$ be a GSS and $s^* \in S'_X$ for some $X \subseteq At$. Then

$$\mathcal{M}, s^* \Downarrow \phi \text{ iff } \phi \in \mathcal{L}^{BA}(X)$$

Proof: see Appendix A.2. ♠

We suggest to keep the underlying GSS, but change the definition of (un)awareness. Semantical *partiality*, stressed above, is an attractive guide. In our introductory example, one would like to say that the possible states that Pierre conceives of do not "answer" the question "Is is true that r ?", whereas they do answer the questions "Is it true that p?" and "Is it true that q ?". In other words, the possible states that Pierre conceives of make neither r nor $\neg r$ true. Awareness can be defined semantically in terms of *partiality* :

$$\mathcal{M}, s \vDash A\phi \text{ iff } \mathcal{M}, \rho(s) \Downarrow \phi$$

Of course, the appeal of this characterization depends on the already given condition given: if $\rho(s) \in S'_X$, then $R(s) \subseteq S'_X$. Let us call a **partial GSS** a GSS where the truth conditions of the (un)awareness operator are in terms of partiality .

Fact 2
Symmetry, Distributivity over $\wedge$, Self-Reflection, U-Introspection, Plausibility and Strong Plausibility are valid under partial GSSs. Furthermore, BU-Introspection is valid under serial partial GSS.

Proof: left to the reader. ♠

5 Probabilistic logic without full awareness

5.1 Language

Definition 6 (probabilistic language with awareness)
The set of formulas of a probabilistic language with awareness $\mathcal{L}^{LA}(At)$ based on a set At of propositional variables is defined by :

$$\phi ::= p|\bot|\top|\neg\phi|\phi \wedge \phi|L_a\phi|A\phi$$

where $p \in At$ and $a \in [0,1] \cap \mathbb{Q}$.

5.2 Generalized Standard Probabilistic Structures

Probabilistic structures make a full awareness assumption, exactly in the same way that Kripke structures do. An obvious way to weaken this assumption is to introduce in the probabilistic setting the same kind of modification as the one investigated in the previous section. The probabilistic counterpart of Generalized Standard Structures are the following Generalized Standard Probabilistic Structures (GSPS):

Definition 7 (generalized standard probabilistic structure)
A generalized standard probabilistic structure for $\mathcal{L}^{LA}(At)$ *is a t-uple* $\mathcal{M} = (S, (S'_X)_{X \subseteq At}, (\Sigma'_X)_{X \subseteq At}, \pi, (P'_X)$
where

- *(i)* S *is a state space.*
- *(ii)* S'_X *where* $X \subseteq At$ *are disjoint "subjective" state spaces. Let* $S' = \bigcup_{X \subseteq At} S'_X$.
- *(ii) For each* $X \subseteq At$, Σ'_X *is a* σ*-field of subsets of* S'_X.
- *(iii)* $\pi : S \times At \rightarrow \{0, 1\}$ *is a valuation.*
- *(iv)* $P'_X : S'_X \rightarrow \Delta(S'_X, \Sigma'_X)$ *is a measurable mapping from* S'_X *to the set of probability measures on* Σ'_X *endowed with the* σ*-field generated by the sets* $\{\mu \in \Delta(S'_X, \Sigma'_X) : \mu(E) \geq a\}$ *for all* $E \in \Sigma'_X$, $a \in [0, 1]$.
- *(v)* $\rho : S \rightarrow S'$ *is an onto map s.t. if* $\rho(s) = \rho(t) \in S'_X$*, then for each atomic formula* $p \in X$, $\pi(s, p) = \pi(t, p)$. *By definition,* $P^*(s^*) = P^*(\rho(s^*))$ *if* $s \in S$ *and* $P^*(s^*) = P'_X(s^*)$ *if* $s^* \in S'_X$.
- *(vi)* $\pi' : S' \times At \rightarrow \{0, 1\}$ *extends* π *to* S' *as follows: for all* $s' \in S'_X$, $\pi'(s', p) = 1$ *iff* $p \in X$ *and for all* $s \in \rho^{-1}(s')$, $\pi(s, p) = 1$[12]. *For every* $p \in At$, $s' \in S'_X$, $\pi'(., p)$ *is measurable w.r.t.* (S'_X, Σ'_X).

Two comments are in order. First, The clause (iv) does not introduce any special measurability condition on the newly introduced awareness operator. (By contrast, there is still a condition for the doxastic operators.) The reason is that in a given subjective state space S'_X, for any formula ϕ, either there is awareness of ϕ everywhere and in this case $[[A\phi]] \cap S'_X = S'_X$, or there is never awareness of ϕ and in this case $[[A\phi]] \cap S'_X = \emptyset$. These two events are of course already in any Σ'_X. Second, clause (v) imposes conditions on the projection ρ. With respect to GSSs, the only change is that we do not require something like: if $\rho(s) \in S'_X$, then $R(s) \subseteq S'_X$. The counterpart would be that if $\rho(s) \in S'_X$, then $Supp(P(s)) \subseteq S'_X$. But this is automatically satisfied by the definition.

<INSERT HERE FIGURE 5>

[12]It follows from clause (v) below that this extension is well defined.

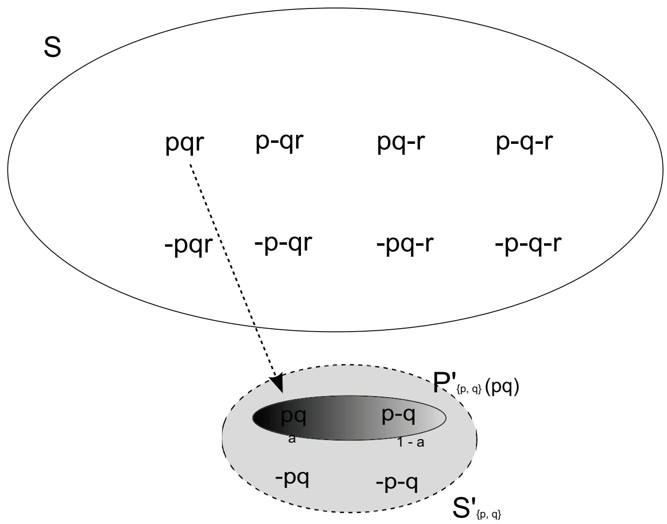


Figure 5: Projection of an objective state to a subjective state space in a probabilistic setting

Example 2
In Figure 5, the support of the probability distribution associated to $s = pqr$ is $\{\rho(s) = pq, p\neg q\}$, $P'_{\{p,q\}}(\rho(s))(pq) = a$ and $P'_{\{p,q\}}(\rho(s))(p\neg q) = 1 - a$.

Definition 8 (satisfaction relation for GSPS)
The ***satisfaction relation*** *for GSPS is defined for each $s^* \in S^* = S \cup S'$:*

(i) $\mathcal{M}, s^* \vDash p$ *iff* $\pi(s^*, p) = 1$

(ii) $\mathcal{M}, s^* \vDash \phi \wedge \psi$ *iff* $\mathcal{M}, s^* \vDash \phi$ *and* $\mathcal{M}, s^* \vDash \psi$

(iii) $\mathcal{M}, s^* \vDash \neg\phi$ *iff* $\mathcal{M}, s^* \nvDash \phi$ *and either* $s^* \in S$, *or* $s^* \in S'_X$ *and* $\phi \in \mathcal{L}^{LA}(X)$

(iv) $\mathcal{M}, s^* \vDash L_a\phi$ *iff* $P^*(s^*)([[\phi]]) \geq a$ *and* $\mathcal{M}, \rho(s) \Downarrow \phi$

(v) $\mathcal{M}, s^* \vDash A\phi$ *iff* $\mathcal{M}, \rho(s^*) \Downarrow \phi$

The following Fact is the counterpart for GSPS of what was proved above for GSS.

Fact 3
Let $\mathcal{M}$ be a GSPS and $s' = \rho(s) \in S'_X$ for some $s \in S$. Then

$$\mathcal{M}, s' \Downarrow \phi \text{ iff } \phi \in \mathcal{L}^{LA}(X)$$

Proof: the proof is analogous to the one provided for Fact 1 above. ♠.

One can show that all the properties mentioned in subsection 2.3 are valid under GSPSs.

Proposition 1
For all GSPS $\mathcal{M}$ and all standard state $s \in S$, the following formulas are satisfied:

$A\phi \leftrightarrow A\neg\phi$	*(Symmetry)*
$A(\phi \wedge \psi) \leftrightarrow A\phi \wedge A\psi$	*(Distributivity over $\wedge$)*
$A\phi \leftrightarrow AA\phi$	*(Self-Reflection)*
$\neg A\phi \rightarrow \neg A\neg A\phi$	*(U-introspection)*
$\neg A\phi \rightarrow \neg L_a\phi \wedge \neg L_a\neg L_a\phi$	*(Plausibility)*
$\neg A\phi \rightarrow (\neg L_a)^n\phi \ \forall n \in \mathbb{N}$	*(Strong Plausibility)*
$\neg L_a\neg A\phi$	*(L_aU-introspection)*
$L_0\phi \leftrightarrow A\phi$	*(Minimality)*

Proof: left to the reader. ♠

5.3 Axiomatization

Proposition 1 suggests that GSPSs provide a plausible analysis of awareness and unawareness in a probabilistic setting. To have a more comprehensive understanding of this model, we need to investigate its logical properties. It turns out that an axiom system can be given that is weakly complete with respect to GSPS. We call it *System HM_U*.

System HM_U

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(A1) $A\phi \leftrightarrow A\neg\phi$
(A2) $A(\phi \wedge \psi) \leftrightarrow A\phi \wedge A\psi$
(A3) $A\phi \leftrightarrow AA\phi$
(A4$_L$) $A\phi \leftrightarrow AL_a\phi$
(A5$_L$) $A\phi \rightarrow L_1A\phi$

(L1$_U$) $A\phi \leftrightarrow L_0\phi$
(L2) $L_a\top$
(L3) $L_a\phi \rightarrow \neg L_b\neg\phi \ (a + b > 1)$
(L4$_U$) $(\neg L_a\phi \wedge A\phi) \rightarrow M_a\phi$

(RE$_U$)From $\phi \leftrightarrow \psi$ and $Var(\phi) = Var(\psi)$[a] infer $(L_a\phi \leftrightarrow L_a\psi)$

(B$_U$) From $((\phi_1, ..., \phi_m) \leftrightarrow (\psi_1, ..., \psi_n))$ infer
$((\bigwedge_{i=1}^{m} L_{ai}\phi_i) \wedge (\bigwedge_{j=2}^{n} M_{bj}\psi_j) \rightarrow (A\psi_1 \rightarrow L_{(a1+...+am)-(b1+...+bn)}\psi_1))$

[a]By definition, $Var(\phi)$ is the set of propositional variables occurring in ϕ.

Some comments are in order. (a) Axioms (A1)-($A5_L$) concern the awareness operator and its relationship with the doxastic operators. The subscript "$_L$" indicates an axiom that involves a probabilistic doxastic operator, to be distinguished from its epistemic counterpart indicated by "$_B$" in the axiom system for epistemic logic reproduced in Appendix A.3. The other axioms and inference rules were roughly part of probabilistic logic without the awareness operator appearing in them. Subscript "$_U$" indicates a modification with respect to the System HM due to the presence of the awareness operator. (b) The relationship between logical omniscience and full awareness come out clearer in the restriction we have to impose on the Rule of Equivalence (RE)[13]. It is easy to see why in semantical terms why the Rule of Equivalence no longer holds universally. Suppose that $L_{1/2}p$ holds at some state of some model. From propositional logic, we know that $p \equiv (p \wedge q) \vee (p \wedge \neg q)$. But if the agent is not aware of q, it is not true that $L_{1/2}((p \wedge q) \vee (p \wedge \neg q))$. (c) For the same kind of reason, the inference rule (B) no longer holds universally. Consider for instance $\phi_1 = (p \vee \neg p)$, $\psi_1 = (q \vee \neg q)$ and $\psi_2 = (p \vee \neg p)$. Clearly, the premiss of (B) i.e. $((\phi_1) \leftrightarrow (\psi_1, \psi_2))$ is satisfied. Furthermore, the antecedent of (B)'s conclusion i.e. $L_1\phi_1$ and $M_1\psi_2$ is satisfied as well. But if the agent is unaware of q, we cannot conclude what we should conclude were (B) valid i.e. that $L_0\psi_1$.

We are now ready to formulate our main result.

Theorem 1 (Soundness and Completeness of HM_U)
Let $\phi \in \mathcal{L}^{LA}(At)$. Then

$$\models_{GSPS} \phi \text{ iff } \vdash_{HM_U} \phi$$

Proof: see the Appendix B. ♠

6 Conclusion

This study of unawareness in probabilistic logic could be unfolded later in several directions. First, we did not deal with the extension to a multi-agent framework - a issue tackled recently by Heifetz et al. (2006). Second, we did not investigate applications to decision theory or game theory. But we would like to end by stressing *another issue* that is less often evoked and nonetheless conceptually very challenging: the dynamics of awareness. The current framework, as much of the existing work, is static, i.e. it captures the awareness and doxastic states at a given time. It does not tell anything on the fact that, during an inquiry, an agent may *become aware* of some new possibilities[14].

Let us consider our initial example where Pierre is aware of p and q but not of r, and let us suppose that Pierre's partial beliefs are represented by some probability distribution

[13]This restricted rule is reminiscent of the rule RE_{sa} in Modica & Rustichini (1999).

[14]Note that he could become *unaware* of some possibilities as well, but we will not say anything about that.

on a subjective state space $S_{\{p,q\}}$. Assume that at some time, Pierre becomes aware of r; for instance, someone has asked him whether he thinks that r is likely or not. It seems that our framework can be extended to accommodate the situation: Pierre's new doxastic state will be represented on a state space $S_{\{p,q,r\}}$ appropriately connected to the initial one $S_{\{p,q\}}$ (see Figure 6). Typically, a state $s' = pq$ will be *split* in two fine-grained states $s_1 = pqr$ and $s_2 = pq\neg r$. But how should Pierre's partial beliefs evolve ? Obviously, a naive Laplacean rule according to which the probability assigned to s' is equally allocated to s_1 and s_2 will not be satisfactory. Are there rationality constraints capable of determining a new probability distribution on $S_{\{p,q,r\}}$? Or should we represent the new doxastic state of the agent by a *set* of probability distributions ? We leave the answers to these questions for future investigation.

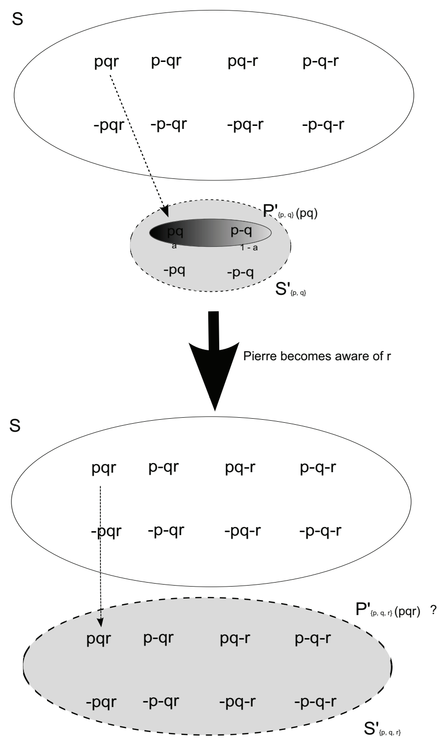


Figure 6: The issue of becoming aware

<INSERT HERE FIGURE 6>

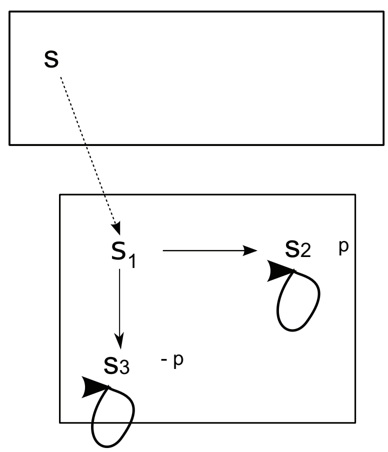
A Epistemic logic: proofs and illustrations

A.1 Illustration

Let us consider a GSS $\mathcal{M}$ where

- the actual state is $s \in S$
- s is projected in $s_1 \in S'_X$ for some $X \subseteq At$
- $R(s_1) = \{s_2, s_3\}$, $R(s_2) = \{s_2\}$ and $R(s_3) = \{s_2\}$
- $\mathcal{M}, s_2 \vDash p$ and $\mathcal{M}, s_3 \vDash \neg p$

The relevant part of the model is represented in the following Figure:



It is easy to check that $\mathcal{M}, s \vDash Up$ and $\mathcal{M}, s \vDash B(B\neg p \vee Bp)$ since $\mathcal{M}, s_2 \vDash Bp$ and $\mathcal{M}, s_3 \vDash B\neg p$.

A.2 Proof of Fact 1

For greater convenience, we give the proof for *partial* GSSs, but Fact 1 holds for original GSSs as well. We have to show that if $\mathcal{M}$ is a partial GSS and $s^* \in S'_X$ for some $X \subseteq At$, then $\mathcal{M}, s^* \Downarrow \phi$ iff $\phi \in \mathcal{L}^{BA}(X)$. The proof is by induction on the complexity of formulas:

- if $\phi := p$, then $\mathcal{M}, s^* \vDash p$ or $\mathcal{M}, s^* \vDash \neg p$ iff $(p \in \mathcal{L}^{BA}(X)$ and $\pi(\rho^{-1}(s^*), p) = 1)$ or $(p \in \mathcal{L}^{BA}(X)$ and not $\pi(\rho^{-1}(s^*), p) = 1)$ iff $p \in \mathcal{L}^{BA}(X)$.

• if $\phi := \neg\psi$, then $\mathcal{M}, s^* \vDash \phi$ or $\mathcal{M}, s^* \vDash \neg\phi$ iff $\mathcal{M}, s^* \vDash \neg\psi$ or $\mathcal{M}, s^* \vDash \neg\neg\psi$ iff ($\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash \psi$) or ($\neg\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash \neg\psi$) iff ($\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash \psi$) or ($\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \vDash \psi$) iff $\psi \in \mathcal{L}^{BA}(X)$ iff $\neg\psi \in \mathcal{L}^{BA}(X)$

• if $\phi := \psi_1 \wedge \psi_2$, then $\mathcal{M}, s^* \vDash \phi$ or $\mathcal{M}, s^* \vDash \neg\phi$ iff ($\mathcal{M}, s^* \vDash \psi_1$ and $\mathcal{M}, s^* \vDash \psi_2$) or ($\psi_1 \wedge \psi_2 \in \mathcal{L}^{BA}(X)$ and ($\mathcal{M}, s^* \nvDash \psi_1$ or $\mathcal{M}, s^* \nvDash \psi_2$)) iff by IH ($\psi_1 \wedge \psi_2 \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \vDash \psi_1$ and $\mathcal{M}, s^* \vDash \psi_2$) or ($\psi_1 \wedge \psi_2 \in \mathcal{L}^{BA}(X)$ and not ($\mathcal{M}, s^* \vDash \psi_1$ and $\mathcal{M}, s^* \vDash \psi_2$)) iff $\psi_1 \wedge \psi_2 \in \mathcal{L}^{BA}(X)$

• if $\phi := B\psi$, then $\mathcal{M}, s^* \vDash \phi$ or $\mathcal{M}, s^* \vDash \neg\phi$ iff (for each $t^* \in R^*(s^*)$, $\mathcal{M}, t^* \vDash \phi$) or ($B\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash B\psi$) iff - by Induction Hypothesis and since each $t^* \in R^*(s^*)$ belongs to S_X - ($B\psi \in \mathcal{L}^{BA}(X)$ $\mathcal{M}, s^* \vDash B\psi$) or ($B\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash B\psi$) iff $B\psi \in \mathcal{L}^{BA}(X)$

• if $\phi := A\psi$, then $\mathcal{M}, s^* \vDash \phi$ or $\mathcal{M}, s^* \vDash \neg\phi$ iff $\mathcal{M}, s^* \Downarrow \psi$ or ($A\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \nvDash A\psi$) iff (by Induction Hypothesis) $\psi \in \mathcal{L}^{BA}(X)$ or ($A\psi \in \mathcal{L}^{BA}(X)$ and $\mathcal{M}, s^* \Uparrow \psi$) iff (by Induction Hypothesis) $\psi \in \mathcal{L}^{BA}(X)$ or ($A\psi \in \mathcal{L}^{BA}(X)$ and $\psi \notin \mathcal{L}^{BA}(X)$) iff $\psi \in \mathcal{L}^{BA}(X)$ iff $A\psi \in \mathcal{L}^{BA}(X)$. ♠

A.3 An axiom system for partial GSSs

We may obtain a complete axiom system for serial partial GSS thanks to Halpern (2001*a*) who relates GSS and awareness© structures. Actually, one obtains a still closer connection with serial partial GSS. Let us first restate the definition of awareness© structures.

Definition 9
*An **awareness© structure** is a t-uple (S, π, R, A) where*

(i) S is a state space,

(ii) $\pi : At \times S \to \{0,1\}$ is a valuation,

(iii) $R \subseteq S \times S$ is an accessibility relation,

(iv) $A : S \to Form(\mathcal{L}^{BA}(At))$ is a function which maps every state in a set of formulas ("awareness© set").

The new condition on the satisfaction relation is the following:
$\mathcal{M}, s \vDash B\phi$ iff $\forall t \in R(t)$ $\mathcal{M}, t \vDash \phi$ and $\phi \in A(s)$
Let us say that an awareness© structure $\mathcal{M} = (S, R, \mathcal{A}, \pi)$ is *propositionally determined* (p.d.) if (1) for each state s, $\mathcal{A}(s)$ is generated by some atomic formulas $X \subseteq At$ i.e. $\mathcal{A}(s) = \mathcal{L}^{BA}(X)$ and (2) if $t \in R(s)$, then $\mathcal{A}(s) = \mathcal{A}(t)$.

Proposition 2 (adapted from Halpern 2001 Thm 4.1)

1. *For every serial p.d. awareness© structure $\mathcal{M}$ there exists a serial partial GSS $\mathcal{M}'$ based on the same state space S and the same valuation π s.t. for all formulas $\phi \in \mathcal{L}^{BA}(At)$ and each possible state s*

   $\mathcal{M}, s \vDash_{a©} \phi$ *iff* $\mathcal{M}', s \vDash_{pGSS} \phi$

2. *For every serial partial GSS $\mathcal{M}$ there exists a serial p.d. awareness*$^{©}$ *structure $\mathcal{M}'$ based on the same state space S and the same valuation π s.t. for all formulas $\phi \in \mathcal{L}^{BA}(At)$ and each possible state s*

   $\mathcal{M}, s \vDash_{pGSS} \phi$ *iff* $\mathcal{M}', s \vDash_{a^{©}} \phi$

Halpern (2001*a*) has devised an axiom system that is (sound and) complete with respect to p.d. awareness$^{©}$ structures. An axiom system for *serial* p.d. awareness$^{©}$ structures can be devised by enriching this axiom system with

$$(\mathrm{D}_U)\ B\phi \rightarrow (\neg B \neg \phi \wedge A\phi)$$

The resulting axiom system coined KD_U is this one:

System KD_U

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(K) $B\phi \wedge B(\phi \rightarrow \psi) \rightarrow B\psi$
(Gen) From ϕ infer $A\phi \rightarrow B\phi$

(D_U) $B\phi \rightarrow (\neg B \neg \phi \wedge A\phi)$
(A1) $A\phi \leftrightarrow A\neg\phi$
(A2) $A(\phi \wedge \psi) \leftrightarrow (A\phi \wedge A\psi)$
(A3) $A\phi \leftrightarrow AA\phi$
($\mathrm{A4}_B$) $AB\phi \leftrightarrow A\phi$
($\mathrm{A5}_B$) $A\phi \rightarrow BA\phi$
(Irr) If no atomic formulas in ϕ appear in ψ, from $U\phi \rightarrow \psi$ infer ψ

The following derives straightforwardly from Proposition 2.

Proposition 3 (Soundness and Completeness Theorem)
Let $\phi \in \mathcal{L}^{BA}(At)$. Then

$$\vDash_{spGSS} \phi \text{ iff } \vdash_{KD_U} \phi$$

B Probabilistic Logic: proof of the Completeness Theorem for HM_U

Proof. ($\Leftarrow$). Soundness is easily checked and is left to the reader. ($\Rightarrow$). We have to show that if $\models_{GSPS} \phi$, then $\vdash_{HM_U} \phi$. The proof relies on the well-known method of *filtration*. First, we define a restricted language $\mathcal{L}^{LA}_{[\phi]}$ as in Heifetz & Mongin (2001) : $\mathcal{L}^{LA}_{[\phi]}$ contains

- as atomic formulas, only $Var(\phi)$ i.e. the atomic formulas occurring in ϕ,
- only probabilistic operators L_a belonging to the finite set $Q(\phi)$ of rational numbers of the form p/q, where q is the smallest common denominator of indexes occurring in ϕ and
- only formulas of epistemic depth smaller than or equal to that of ϕ. (An important point is that we stipulate that the awareness operator A does not add any epistemic depth to a formula: $dp(A\psi) = dp(\psi)$.)

As we will show, the resulting language $\mathcal{L}^{LA}_{[\phi]}$ is *finitely generated*: there is a finite subset $\mathfrak{B}$ of $\mathcal{L}^{LA}_{[\phi]}$ called a *base* such that $\forall \psi \in \mathcal{L}^{LA}_{[\phi]}$ there is a formula ψ' in the base such that $\vdash_{HM_U} \psi \leftrightarrow \psi'$. In probabilistic structures, it is easy to construct such a base[15]. The basic idea is this:

(1) consider D_0, the set of all the disjunctive normal forms built from $B_0 = Var(\phi)$, the set of propositional variables occurring in ϕ.

(2) B_k is the set of formulas $L_a\psi$ for all $a \in Q(\phi)$ where ψ is a disjunctive normal form built with "atoms" coming from B_0 to B_{k-1}.

(3) the construction has to be iterated up to the epistemic depth n of ϕ hence to B_n. The base $\mathfrak{B}$ is D_n, i.e. the set of disjunctive normal forms built with "atoms" from B_0 to B_n.

Obviously, $\mathfrak{B}$ is finite. It can be shown by induction that $\mathcal{L}^{LA}_{[\phi]}$ is finitely generated by $\mathfrak{B}$. For formulas with a Boolean connective as top connective, this is obvious. For formulas of the form $L_a\psi$, it comes from substituability under logically equivalent formulas : by induction hypothesis, there is a formula ψ' equivalent to ψ in $\mathfrak{B}$. Therefore, there is in $\mathfrak{B}$ a formula equivalent to $L_a\psi'$. But since $\vdash_{HM} \psi \leftrightarrow \psi'$, it follows that $\vdash_{HM} L_a\psi \leftrightarrow L_a\psi'$. We will now show how to unfold formally these ideas.

Definition 10

Let $X = \{\psi_1, ...\psi_m\}$ be a finite set of formulas. $DNF(X)$ is the set of Disjunctive Normal Forms that can be built from X i.e. the set of all possible disjunctions of conjunctions of the form $e_1\psi_1 \wedge ... \wedge e_m\psi_m$ where e_i is a blank or $\neg$. The members of X are called the atomic components of $DNF(X)$.

[15] Heifetz & Mongin (2001) leave the construction implicit.

Definition 11
The base $\mathfrak{B}$ for a language $\mathcal{L}^L_{[\phi]}$ where $dp(\phi) = n$ is defined as D_n in the following doubly inductive construction:

(i) $B_0 = Var(\phi)$ (B_0 is the set of atomic components of epistemic depth 0)
(i') $D_0 = DNF(B_0)$ (D_0 is the set of disjunctive normal forms based on B_0)

(ii) $B_k = \{L_a\psi : \psi \in D_{k-1}\}$
(ii') $D_k = DNF(\bigcup_{l=0}^{k} B_l)$.

Notation 1
Let $\psi \in DNF(X)$ and $X \subseteq Y$. The expansion of ψ in $DNF(Y)$ is the formula obtained by the replacement of each conjunction $e_1\psi_1 \wedge ... \wedge e_m\psi_m$ occurring in ψ by a disjunction of all possible conjunctions built from $e_1\psi_1 \wedge ... \wedge e_m\psi_m$ by adding literals of atomic components in $Y - X$.

For instance, consider $X = \{p\}$ and $Y = \{p, q\}$: the DNF p is expanded in $(p \wedge q) \vee (p \wedge \neg q)$.

Fact 4
(i) $\forall k$, $\forall \psi \in D_k \cup B_k$, $dp(\psi) = k$.
(ii) For each $\psi \in D_k$, in each D_l, $l > k$, there is a formula ψ' which is equivalent to ψ

Proof: (i) is obvious. (ii) follows from the fact that any formula $\psi \in D_k$ can be expanded in ψ' from D_l, $l > k$ and that (by propositional reasoning), ψ and ψ' are equivalent. ♠

It can be proven that $\mathfrak{B} = D_n$ is a finite base for $\mathcal{L}^L_{[\phi]}$. First, for each $\psi \in \mathcal{L}^L_{[\phi]}$ there is a formula ψ' in D_l s.t. $\vdash_{HM} \psi \leftrightarrow \psi'$ where $dp(\psi) = l$. Since ψ' can be expanded in a logically equivalent formula $\psi'' \in D_n$, it is sufficient to conclude that for each $\psi \in \mathcal{L}^L_{[\phi]}$ there is an equivalent formula in the base.

(i) $\psi := p$: ψ is obviously equivalent to some DNF in D_0 and $dp(p) = 0$.

(ii) $\psi := (\chi_1 \wedge \chi_2)$: by Induction Hypothesis there is χ'_1 equivalent to χ_1 in $D_{dp(\chi_1)}$ and χ'_2 equivalent to χ_2 in $D_{dp(\chi_2)}$. Suppose w.l.o.g. that $dp(\chi_2) > dp(\chi_1)$ and therefore that $dp(\psi) = dp(\chi_2)$. Then χ'_1 can be expanded in $\chi''_1 \in D_{dp(\chi_2)}$. Obviously, the disjunction of the conjunctions occurring both in χ''_1 and χ'_2 is in $D_{dp(\chi_2)}$ and equivalent to ψ.

(iii) $\psi := L_a\chi$: by IH, there is χ' equivalent to χ in $D_{dp(\chi)}$. Note that $dp(\chi) < n = dp(\phi)$. By construction, $L_a(\chi') \in B_{dp(\chi)+1}$. Consequently, there will be in $D_{dp(\chi)+1}$ a DNF ψ' equivalent to $L_a(\chi')$. Since $dp(\chi) + 1 \leq n$, this DNF can be associated by expansion to a DNF in the base D_n. Furthermore, since $\vdash_{HM} \chi \leftrightarrow \chi'$ and $\vdash_{HM} L_a\chi' \leftrightarrow \psi'$, it follows by the Rule of Equivalence that $\vdash_{HM} L_a\chi \leftrightarrow \psi'$. ♠

There changes are needed to deal with unawareness. (1) First, the awareness operator A has to be included. This is not problematic given that for any formula ψ, $\vdash_{HM_U} A\psi \leftrightarrow \bigwedge_m Ap_m$ where $Var(\psi) = \{p_1, ..., p_m, ...p_M\}$. Consequently, the only modification is to include any formula Ap with $p \in Var(\phi)$ in B_0. (2) With unawareness, it is no longer true that if $\vdash_{HM_U} \psi \leftrightarrow \chi$, then $\vdash_{HM_U} L_a\psi \leftrightarrow L_a\chi$. For instance, it is not true that $\vdash_{HM_U} L_a p \leftrightarrow L_a((p \wedge q) \vee (p \wedge \neg q))$: the agent may be unaware of q. Nevertheless, the rule holds restrictedly: under the assumption that $Var(\psi) = Var(\chi)$, then if $\vdash_{HM_U} \psi \leftrightarrow \chi$, then $\vdash_{HM_U} L_a\psi \leftrightarrow L_a\chi$ (RE$_U$). We can use this fact to make another change to the basis: instead of considering only the disjunctive normal forms built from the whole set $Var(\phi)$, we consider the disjunctive normal forms built from *any non-empty subset* $X \subseteq Var(\phi)$.

Definition 12
Let $X \subseteq Var(\phi)$*;*
(i) $B_0^X = X \cup \{Ap : p \in X\}$
(i') $D_0^X = DNF(B_0)$

(ii) $B_k^X = \{L_a\psi : \psi \in D_{k-1}\}$*, and*
(ii') $D_k^X = DNF(\bigcup_{l=0}^{k} B_l^X)$*.*

Fact 5
(i) $\forall k \leq n$, $\psi \in D_k^X$ *where* $X \subseteq Var(\phi)$, $dp(\psi) = k$*.*
(ii) $\forall X \subseteq Var(\phi)$, $\forall \psi \in D_k^X$, $\forall D_l^X$, $l > k$*, there is a formula* $\psi' \in D_l^X$ *which is equivalent to* ψ*.*
(iii) $\forall X \subseteq Y \subseteq Var(\phi)$*, if* $\psi \in D_k^X$ *then there is a formula* ψ' *which is equivalent to* ψ *in* D_k^Y*.*
(iv) $\forall X \subseteq Var(\phi)$, $\forall \psi \in D_k^X$, $Var(\psi) = X$*.*

Proof: (i)-(ii) are similar to the classical case. (iii) is straightforwardly implied by the clause (ii') of the Definition 12. (iv) is obvious. ♠

We are now ready to prove that $\mathfrak{B} = \bigcup_{X \subseteq Var(\phi)} D_n^X$ is a basis for $\mathcal{L}_{[\phi]}^{LA}$. We will actually show that for any $\psi \in L_{[\phi]}^{LA}$ with $dp(\psi) = k$, there are $X \subseteq Var(\phi)$ and $\psi' \in D_k^X$ s.t. $\vdash_{HM_U} \psi \leftrightarrow \psi'$ and $dp(\psi) = dp(\psi') = k$ and $Var(\psi) = Var(\psi')$ (IH).

(i) $\psi := p$: ψ is obviously equivalent to some DNF ψ' in $D_0^{\{p\}}$. Clearly $dp(\psi) = dp(\psi')$ and $Var(\psi) = Var(\psi')$.

(ii) $\psi := (\chi_1 \wedge \chi_2)$: by IH,

- there is χ_1' s.t. $\vdash_{HM_U} \chi_1 \leftrightarrow \chi_1'$ and $Var(\chi_1) = Var(\chi_1') = X_1$ and $\chi_1' \in D_{dp(\chi_1)}^{X_1}$
- there is χ_2' s.t. $\vdash_{HM_U} \chi_2 \leftrightarrow \chi_2'$ and $Var(\chi_2) = Var(\chi_2') = X_2$ and $\chi_2' \in D_{dp(\chi_2)}^{X_2}$

Let us consider $X' = X_1 \cup X_2$ and suppose w.l.o.g. that $dp(\chi_2) > dp(\chi_1)$. One may expand χ_1' from $D_{dp(\chi_1)}^{X_1}$ to $D_{dp(\chi_1)}^{X'}$ and expand the resulting DNF to $\chi_1'' \in D_{dp(\chi_2)}^{X'}$. On the other hand, χ_2' may be expanded to $\chi_2'' \in D_{dp(\chi_2)}^{X'}$. ψ' is the disjunction of the conjunctions common to χ_1'' and χ_2''. Obviously, $dp(\psi) = dp(\psi')$ and $Var(\psi) = Var(\psi')$.

(iii) $\psi := A\chi$: by IH, there is χ' equivalent to χ in $D^X_{dp(\chi)}$ with $Var(\chi) = Var(\chi')$. $A\chi'$ is equivalent to $\bigwedge_m Ap_m$ where $Var(\chi') = \{p_1, ..., p_m, ...p_M\}$. Each Ap_m is in B^X_0, so by expansion in $D^X_{dp(\chi)}$ there is a DNF equivalent to it and therefore a DNF equivalent to $\bigwedge_m Ap_m$.

(iv) $\psi := L_a\chi$: by IH, there is χ' equivalent to χ in $D^X_{dp(\chi)}$ with $dp(\chi) = dp(\chi')$ and $Var(\chi) = Var(\chi')$. Note that $dp(\chi) < n = dp(\phi)$. By construction, $L_a(\chi') \in B^X_{dp(\chi)+1}$. Consequently, there will be in $D^X_{dp(\chi)+1}$ a DNF ψ' logically equivalent to $L_a(\chi')$. Since $dp(\chi)+1 \leq n$, there will be in the base a formula ψ'' logically equivalent to ψ'. Furthermore, since $\vdash_{HM_U} \chi \leftrightarrow \chi'$ and $Var(\chi) = Var(\chi')$ and $\vdash_{HM_U} L_a\chi' \leftrightarrow \psi''$, it follows that $\vdash_{HM_U} L_a\chi \leftrightarrow \psi''$. ♠

We will now build (1) the objective state space, (2) the subjective states spaces and the projection ρ, and (3) the probability distributions.

(1) The objective state space

The objective states of the ϕ-canonical structure are the intersections of the maximally consistent sets of formulas of the language $\mathcal{L}^{LA}(At)$ and the restricted language $\mathcal{L}^{LA}_{[\phi]}$:

$$S^\phi = \{\Gamma \cap \mathcal{L}^{LA}_{[\phi]} : \Gamma \text{ is a maximal } HM_U\text{-consistent set}\}$$

First, let's notice that the System HM_U is a "modal logic" (Blackburn, de Rijke & Venema (2001), p. 191): a set of formulas (1) that contains every propositional tautologies, (2) such that the Lindenbaum Lemma holds.

Definition 13

(i) A formula ϕ is ***deducible*** *from a set of formulas Γ, symbolized $\Gamma \vdash_{HM_U} \phi$, if there exists some formulas $\psi_1, ..., \psi_n$ in Γ s.t. $\vdash_{HM_U} (\psi_1 \wedge ... \wedge \psi_n) \to \phi$.*

*(ii) A set of formulas Γ is HM_U-****consistent*** *if it is false that $\Gamma \vdash_{HM_U} \bot$*

(iii) A set of formulas Γ is ***maximally*** *HM_U-****consistent*** *if (1) it is HM_U-consistent and (2) if it is not included in a HM_U-consistent set of formulas.*

Lemma 1 (Lindenbaum Lemma)

If Γ is a set of HM_U-consistent formulas, then there exists an extension Γ^+ of Γ that is maximally HM_U-consistent.

Proof: See for instance Blackburn et al. (2001), p.199. ♠

Notation 2

For each formula $\psi \in \mathcal{L}^{LA}_{[\phi]}$, let us note $[\psi] = \{s \in S^\phi : \psi \in s\}$

Lemma 2

The set S^ϕ is finite.

Proof: This Lemma is a consequence of the fact that $\mathcal{L}^{LA}_{[\phi]}$ is *finitely generated*.

(a) Let us say that two sets of formulas are Δ-equivalent if they agree on each formula that belongs to Δ. S^{ϕ} identifies the maximal HM_U-consistent sets of formulas that are $\mathcal{L}^{LA}_{[\phi]}$-equivalent. S^{ϕ} is infinite iff there are infinitely many maximal HM_U-consistent sets of formulas that are *not* pairwise $\mathcal{L}^{LA}_{[\phi]}$-equivalent.

(b) If $\mathfrak{B}$ is a base for $\mathcal{L}^{LA}_{[\phi]}$, then two sets of formulas are $\mathcal{L}^{LA}_{[\phi]}$-equivalent iff they are $\mathfrak{B}$-equivalent. Suppose that Δ_1 and Δ_2 are not $\mathcal{L}^{LA}_{[\phi]}$-equivalent. This means w.l.o.g. that there is a formula ψ s.t. (i) $\psi \in \Delta_1$, (ii) $\psi \notin \Delta_2$ and (iii) $\psi \in \mathcal{L}^{LA}_{[\phi]}$. Let $\psi' \in \mathfrak{B}$ a formula s.t. $\vdash_{HM_U} \psi \leftrightarrow \psi'$. Clearly, $\psi' \in \Delta_1$ and $\psi' \in \mathcal{L}^{LA}_{[\phi]}$ and $\neg\psi' \in \Delta_2$. Therefore Δ_1 and Δ_2 are not $\mathfrak{B}$-equivalent. The other direction is obvious.

(c) Since $\mathfrak{B}$ is finite, there are only finitely many maximal HM_U-consistent sets of formulas that are *not* pairwise $\mathfrak{B}$-equivalent. So S^{ϕ} is finite. ♠

(2) The subjective state spaces and the projection $\rho(.)$

As it might be expected, the subjective state associated with an objective state $\Gamma \in S^{\phi}$ will be determined by the formulas that the agent is aware of in Γ.

Definition 14
For any set of formulas Γ, let $Var(\Gamma)$ be the set of atomic formulas that occur in the formulas that belong to Γ. For any $\Gamma \in S^{\phi}$, let

(i) $A^+(\Gamma) = \{\psi : A\psi \in \Gamma\}$ *and* $A^-(\Gamma) = \{\psi : \neg A\psi \in \Gamma\}$

(ii) $a^+(\Gamma) = \{p \in Var(\mathcal{L}^{LA}_{[\phi]}) : Ap \in \Gamma\}$ *and* $a^-(\Gamma) = \{p \in Var(\mathcal{L}^{LA}_{[\phi]}) : \neg Ap \in \Gamma\}$.

Lemma 3
Let $\Gamma \in S^{\phi}$.

(i) $A^+(\Gamma) = \mathcal{L}^{LA}_{[\phi]}(a^+(\Gamma))$

(ii) $A^+(\Gamma) \cup A^-(\Gamma) = \mathcal{L}^{LA}_{[\phi]}$

Proof: (i) Follows from (A1)-(A4$_L$). (ii) Follows from (i) and the fact that since Γ comes from a maximal consistent set, $\neg\psi \in \Gamma$ iff $\psi \notin \Gamma$. ♠.
One may group the sets that have the same awareness profile into equivalence classes: $|\Gamma|_a = \{\Delta \in S^{\phi}\text{: } a^+(\Delta) = a^+(\Gamma)\}$. The sets that belong to the same equivalence class $|\Gamma|_a$ will be mapped in the same subjective state space $S'_{|\Gamma|_a}$. We are now ready to define the projection ρ and these subjective states.

Definition 15
The projection $\rho : S^{\phi} \to \bigcup_{\Gamma \in S^{\phi}} S'_{|\Gamma|_a}$ *is defined by*

$$\rho(\Gamma) = \Gamma \cap A^+(\Gamma)$$

where $S'_{|\Gamma|_a} = \{\Delta \cap A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]} : \Delta$ *is a maximal* HM_U*-consistent set and* $a^+(\Delta) = a^+(\Gamma)\}$*.*

Note that in the particular case where the agent is unaware of every formula, $A^+(\Gamma) = \emptyset$. So each objective state where the agent is unaware of every formula will be projected in the same subjective state $\emptyset \in S'_{\emptyset} = \{\emptyset\}$. More importantly, one can check that ρ is an onto map: suppose that $\Lambda \in S'_{|\Gamma|_a}$ where $\Gamma \in S^{\phi}$. By definition, for some Δ (a maximal HM_U-consistent set), $\Lambda = \Delta \cap A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$ and $a^+(\Delta) = a^+(\Gamma)$. As a consequence, $A^+(\Delta) = A^+(\Gamma)$ and therefore $\Lambda = \Delta \cap A^+(\Delta) \cap \mathcal{L}^{LA}_{[\phi]}$. Hence $\Lambda = \rho(\Delta \cap \mathcal{L}^{LA}_{[\phi]})$. One can show also the following Lemma.

Lemma 4

(i) For each $\Gamma \in S^{\phi}$*,* $S'_{|\Gamma|_a}$ *is finite.*

(ii) For each subset $E \subseteq S'_{|\Gamma|_a}$ *there is* $\psi \in A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$ *s.t.* $E = [\psi]_{S'_{|\Gamma|_a}}$ *where* $[\psi]_{S'_{|\Gamma|_a}}$ *denotes the set of states of* $S'_{|\Gamma|_a}$ *to which* ψ *belongs.*

(iii) For all $\psi_1, \psi_2 \in A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$*,* $[\psi_1]_{S'_{|\Gamma|_a}} \subseteq [\psi_2]_{S'_{|\Gamma|_a}}$ *iff* $\vdash_{HM_U} \psi_1 \rightarrow \psi_2$

Proof: (i) Follows trivially since the objective state space is already finite. (ii) Let us pick a finite base $\mathfrak{B}_\Gamma$ for $A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$. For each element β of this base and each $\Delta \in S'_{|\Gamma|_a}$, either $\beta \in \Delta$ or $\neg\beta \in \Delta$. Two sets Δ and $\Delta' \in S'_{|\Gamma|_a}$ differ at least by one such formula of $\mathfrak{B}_\Gamma$. Let $C(\Delta) = \bigwedge_m e_m \beta_m$ where $\beta_m \in \mathfrak{B}_\Gamma$ and e_m is a blank if $\beta_m \in \Delta$ and $\neg$ if $\beta_m \notin \Delta$. For two distinct sets Δ and Δ', $C(\Delta) \neq C(\Delta')$. For each event $E \subseteq S'_{|\Gamma|_a}$, one can therefore consider the disjunction $\bigvee_k C(\Delta_k)$ for each $\Delta_k \in E$. Such a formula belongs to each Δ_k and only to these Δ_k. (iii) ($\Rightarrow$). For each formula $\psi \in A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$ and each $\Delta \in S'_{|\Gamma|_a}$, $\neg\psi \in \Delta$ iff $\psi \notin \Delta$. Therefore, there are two possibilities for any Δ : either $\psi \in \Delta$ or $\neg\psi \in \Delta$. (a) If $\psi_1 \in \Delta$, then by hypothesis $\psi_2 \in \Delta$ and given the construction of the language, $\neg\psi_1 \vee \psi_2 \in \Delta$ hence $\psi_1 \rightarrow \psi_2 \in \Delta$. (b) If $\psi_1 \notin \Delta$, then $\neg\psi_1 \in \Delta$ hence $\psi_1 \rightarrow \psi_2 \in \Delta$. This implies that for any Δ, $\psi_1 \rightarrow \psi_2 \in \Delta$. Given the definition of $S'_{|\Gamma|_a}$ and the properties of maximal consistent sets, this implies that $\vdash_{HM_U} \psi_1 \rightarrow \psi_2$. ($\Leftarrow$). Given the construction of the language, if $\psi_1, \psi_2 \in A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$, then $\psi_1 \rightarrow \psi_2 \in A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]}$. Since $\vdash_{HM_U} \psi_1 \rightarrow \psi_2$, for each Δ, $\psi_1 \rightarrow \psi_2 \in \Delta$. If $\psi_1 \in \Delta$, clearly $\psi_2 \in \Delta$ as well. Therefore $[\psi_1]_{S'_{|\Gamma|_a}} \subseteq [\psi_2]_{S'_{|\Gamma|_a}}$. ♠

(3) The probability distributions

Definition 16

For $\Gamma \in S^{\phi}$ *and* $\psi \in \mathcal{L}^{LA}_{[\phi]}$*, let*

- $\tilde{a} = \max\{a : L_a\psi \in \Gamma\}$
- $\tilde{b} = \min\{b : M_b\psi \in \Gamma\}$

In the classical case (Heifetz & Mongin 2001), $\tilde{a}$ and $\tilde{b}$ are always defined. This is not so in our structure with unawareness: if the agent is not aware of ψ, no formula $L_a\psi$ will be true because of ($A0_U$) $A\psi \leftrightarrow L_0\psi$. Given (A1) and (DefM), one can derive

$\vdash_{HM_U} A\psi \leftrightarrow M_1\psi$

The construction of the language implies that for any Γ, $A\psi \in \Gamma$ iff $L_0\psi \in \Gamma$ iff $M_1\psi \in \Gamma$. Therefore $\tilde{a}$ and $\tilde{b}$ are defined iff $A\psi \in \Gamma$.

Lemma 5

Let us suppose that $A\psi \in \Gamma$.

(i) $\forall c \in Q(\phi)$, *$c \leq \tilde{a}$ implies $L_c\psi \in \Gamma$ and $c \geq \tilde{b}$ implies $M_c\psi \in \Gamma$*

(ii) There are only two cases : (i) either $\tilde{a} = \tilde{b}$ and $E_{\tilde{a}}\psi \in \Gamma$ while $E_c\psi \notin \Gamma$ for $c \neq \tilde{a}$, (ii) or $\tilde{a} < \tilde{b}$ and $E_c\psi \notin \Gamma$ for any $c \in Q(\phi)$.

(iii) $\tilde{b} - \tilde{a} \leq \frac{1}{q}$ *(where q is the common denominator to the indexes)*

Proof: see Heifetz & Mongin (2001), the modifications are obvious. ♠

Definition 17

Given $\Gamma \in S^\phi$ and $\psi \in \mathcal{L}^{LA}_{[\phi]}$, if $A\psi \in \Gamma$, let

I^Γ_ψ be either $\{\tilde{a}\}$ if $\tilde{a} = \tilde{b}$ or $(\tilde{a}, \tilde{b})$ if $\tilde{a} < \tilde{b}$.

Heifetz & Mongin (2001)'s Lemma A.5 can be adapted to show that for each $S'_{|\Gamma|_a}$ and $\Gamma \in S'_{|\Gamma|_a}$, there is a probability distribution $P'_{|\Gamma|_a}(\Gamma)$ on $2^{S'_{|\Gamma|_a}}$ such that

(C) for all $\psi \in \mathcal{L}^{LA}_{[\phi]}$ if $A\psi \in \Gamma$, $P'_{|\Gamma|_a}(\Gamma)([\psi]_{S'_{|\Gamma|_a}}) \in I^\Gamma_\psi$.

Heifetz & Mongin (2001)'s proof relies on a theorem by Rockafellar that can be used because of the inference rule (B). It would be tedious to adapt the proof here. One comment is nonetheless important. In our axiom system HM_U, the inference rule holds under a restricted form (B_U). So one could wonder whether this will not preclude adaptating the original proof which relies on the unrestricted version (B). It turns out that the answer is negative. The reason is that the formulas involved in the application of (B) are only representatives for each subset of the state space. We have previously shown how to build these formulas in our case, and they are such that the agent is necessarily aware of them. So the restriction present in (B_U) does not play any role, and we may define the ϕ-canonical structure as follows.

Definition 18

*The ϕ-**canonical structure** is the GSPS $\mathcal{M}_\phi = (S^\phi, S^{\phi\prime}, (2^{S'_{|\Gamma|_a}})_{\Gamma \in S^\phi}, \pi_\phi, (P^\phi_{|\Gamma|_a})_{\Gamma \in S^\phi})$ where*

(i) $S^\phi = \{\Gamma \cap \mathcal{L}^{LA}_{[\phi]} : \Gamma$ *is a maximal HM_U-consistent set*$\}$

(ii) $S^{\phi'} = \bigcup_{\Gamma \in S^\phi} S'_{|\Gamma|_a}$ *where* $S'_{|\Gamma|_a} = \{\Delta \cap A^+(\Gamma) \cap \mathcal{L}^{LA}_{[\phi]} : \Delta$ *is a maximal* HM_U*-consistent set and* $a(\Delta) = a(\Gamma)\}$

(iii) *for each* $\Gamma \in S^\phi$, $\rho(\Gamma) = \Gamma \cap A^+(\Gamma)$

(iv) *for all state* $\Gamma \in S^\phi \cup S^{\phi'}$ *and atomic formula* $p \in At$, $\pi_\phi(p, \Gamma) = 1$ *iff* $p \in \Gamma$

(v) *for* $\Gamma \in S^\phi$, $P^\phi_{|\Gamma|_a}$ *is a probability distribution on* $2^{S'_{|\Gamma|_a}}$ *satisfying condition (C)*[16].

We are now ready to state the crucial Truth Lemma.

Lemma 6 (Truth Lemma)
For every $\Gamma \in S^\phi$ *and every* $\psi \in, \mathcal{L}^{LA}_{[\phi]}$,

$$\mathcal{M}_\phi, \Gamma \vDash \psi \textit{ iff } \psi \in \Gamma$$

Proof: The proof proceeds by induction on the complexity of the formula.

- $\psi := p$; follows directly from the definition of π_ϕ.
- $\psi := \neg\chi$. Since Γ is a standard state $\mathcal{M}_\phi, \Gamma \vDash_\phi \neg\chi$ iff $\mathcal{M}_\phi, \Gamma \nvDash \chi$ iff (by IH) $\chi \notin \Gamma$. We shall show that $\chi \notin \Gamma$ iff $\neg\chi \in \Gamma$. ($\Rightarrow$) Let us suppose that $\chi \notin \Gamma$; χ is in $\mathcal{L}^{LA}_{[\phi]}$ hence, given the properties of maximally consistent sets, $\neg\chi \in \Gamma^+$ where Γ^+ is the extension of Γ to $\mathcal{L}^{LA}(At)$ (the whole language). And since $\Gamma = \Gamma^+ \cap \mathcal{L}^{LA}_{[\phi]}$, $\neg\chi \in \Gamma$. ($\Leftarrow$) Let us suppose that $\neg\chi \in \Gamma$. Γ is coherent, therefore $\chi \notin \Gamma$.
- $\psi := \psi_1 \wedge \psi_2$. ($\Rightarrow$). Let us assume that $\mathcal{M}_\phi, \Gamma \vDash \psi_1 \wedge \psi_2$. Then $\mathcal{M}_\phi, \Gamma \vDash \psi_1$ and $\mathcal{M}_\phi, \Gamma \vDash \psi_2$. By IH, this implies that $\psi_1 \in \Gamma$ and $\psi_2 \in \Gamma$. Given properties of maximally consistent sets, this implies in turn that $\psi_1 \wedge \psi_2 \in \Gamma$. ($\Leftarrow$). Let us assume that $\psi_1 \wedge \psi_2 \in \Gamma$. Given the properties of maximally consistent sets, this implies that $\psi_1 \in \Gamma$ and $\psi_2 \in \Gamma$ and therefore, by IH, that $\mathcal{M}_\phi, \Gamma \vDash \psi_1$ and $\mathcal{M}_\phi, \Gamma \vDash \psi_2$.
- $\psi := A\chi$. We know that in any GSPS $\mathcal{M}$, if $s' = \rho(s) \in S'_X$ for some $s \in S$, then $\mathcal{M}, s' \Downarrow \chi$ iff $\chi \in \mathcal{L}^{LA}(X)$. In our case, $s = \Gamma$, $s' = \rho(\Gamma)$ and $X = a^+(\Gamma)$. So $\mathcal{M}_\phi, \Gamma \vDash A\chi$ iff $\chi \in \mathcal{L}^{LA}(a^+(\Gamma))$. But given that $A\chi \in \mathcal{L}^{LA}_{[\phi]}$, $\chi \in \mathcal{L}^{LA}(a^+(\Gamma))$ iff $A\chi \in \Gamma$.
- $\psi := L_a\chi$. By definition $\mathcal{M}_\phi, \Gamma \models L_a\chi$ iff $P_{|\rho(\Gamma)|_a}(\Gamma)([[\chi]]) \geq a$ *and* $\mathcal{M}_\phi, \rho(\Gamma) \Downarrow \chi$.($\Leftarrow$) Let us suppose that $P_{|\rho(\Gamma)|_a}(\Gamma)([[\chi]]) \geq a$ and $\mathcal{M}_\phi, \rho(\Gamma) \Downarrow \chi$. Hence $\tilde{a}$ is well-defined. It is clear that $\tilde{a} \geq a$ given our definition of $P_{|\rho(\Gamma)|_a}(\Gamma)$. It is easy to see that $\vdash_{HM_U} L_a\psi \rightarrow L_b\psi$ for $b \leq a$. As a consequence, $L_a\psi \in \Gamma$. ($\Rightarrow$) Let us suppose that $L_a\psi \in \Gamma$. This implies that $A\psi \in \Gamma$ and therefore that $\mathcal{M}_\phi, \rho(\Gamma) \Downarrow \chi$. By construction, $a \leq \tilde{a}$ and therefore $P_{|\rho(\Gamma)|_a}(\Gamma)([[\chi]]) \geq a$. Hence $\mathcal{M}_\phi, \Gamma \models L_a\chi$. ♠

[16]In the particular case where $A^+(\Gamma) = \emptyset$, the probability assigns maximal weight to the only state of $S'_\emptyset$.

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